THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2023 Enrichment Programme for Young Mathematics Talents
SAYT1134 Towards Differential Geometry
Test 2
Time allowed: $90 \pm \delta$ **Minutes**

Instructions

- Time allowed: $90 \pm \delta$ minutes.
- Show your work clearly and concisely.
- Give adequate explanation and justification for your calculation and observation.
- Write your answers in the spaces provided.
- Supplementary answer sheets and rough paper will be supplied on request. Write your full name in English and indicate the question number attempting (if applicable) on the top of each sheet.
- Unless otherwise specified, numerical answers must be exact.
- Calculators are not allowed.
- The highest attainable score of this paper is 100, and it contributes 30% towards your final score.
- You are reminded that marks will be given for partial attempts.
- Good luck!

Full Name in English: ____________________ Group: ____________________

Question	Points	Bonus Points	Score
1	30	0	
2	16	0	
3	14	0	
4	15	0	
5	12	0	
6	13	0	
7	0	15	
Total:	100	15	

Formula Sheet

1. For a plane curve $\mathbf{r}(t) = (x(t), y(t))$, we have
$$\kappa(t) = \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{\frac{3}{2}}}.$$

2. For a space curve $\mathbf{r}(t) = (x(t), y(t), z(t))$, we have
$$\kappa(t) = \frac{||\mathbf{r}' \times \mathbf{r}''||}{||\mathbf{r}'||^3}.$$

1. Compute the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, curvature κ and torsion τ of the space curves below.
   (a) (18 points) $\alpha(s) = \left(\frac{4}{9}(1+s)^{\frac{3}{2}}, \frac{4}{9}(1-s)^{\frac{3}{2}}, \frac{1}{3}s\right), s \in (-1, 1)$.
   (b) (12 points) $\alpha(\theta) = \left(6\cos 2\theta \cos^3\left(\frac{2\theta}{3}\right),\ 6\sin 2\theta \cos^3\left(\frac{2\theta}{3}\right),\ \frac{1}{2}\cos 4\theta - \cos^2 2\theta\right),\ \theta \in \left(0, \frac{\pi}{4}\right)$.

2. Let $\mathbf{r} : (0, \ln 2) \to \mathbb{R}^3$ be a curve defined by $\mathbf{r}(t) = (\cosh t, \sinh t, t)$.
   (a) (2 points) Suppose $x \in \mathbb{R}$. Show that there exists exactly one $t \in \mathbb{R}$ such that $\sinh t = x$, and express t in terms of x.
   (b) (10 points) Let $\mathbf{r}_1(s)$ be the arc-length parameterization of the curve. Find an explicit formula for $\mathbf{r}_1(s)$. State clearly the range of possible values of the parameter s.
   (c) (4 points) Compute the tangent vector $\mathbf{T}(s_0)$ of $\mathbf{r}_1(s)$ at $s = s_0$.

3. The logarithmic spiral is a curve defined by $r = e^{\theta}$ in polar coordiantes.
   (a) (6 points) Find the arc-length of the logarithmic spiral from $\theta = 0$ to $\theta = 2\pi$.
   (b) (8 points) Find the curvature of the logarithmic spiral.

4. (15 points) Let $\mathbf{r}(t)$ be a regular parametrized space curve with $\kappa(t) > 0$ for any t. Denote the torsion at $\mathbf{r}(t)$ by $\tau(t)$. Prove that $\mathbf{r}(t)$ is contained in a plane if and only if $\tau(t) = 0$ for any t.
(Hint: A space curve $\mathbf{r}$ is contained in a plane if there exists a fixed unit vector $\mathbf{n}$ such that $\langle \mathbf{r}, \mathbf{n} \rangle$ is a constant.)

5. (12 points) Let $\alpha(s) : I \to \mathbb{R}^2$ be a regular arc length parametrized plane curve. Suppose that $\mathbf{p} \in \mathbb{R}^2$ is a point such that $\alpha(s) \neq \mathbf{p}$ for all $s \in I$. Suppose there exists $s_0 \in I$ such that

$$\|\alpha(s_0) - \mathbf{p}\| = \max_{s \in I}\{\|\alpha(s) - \mathbf{p}\|\}.$$

Denote the curvature of α at $s = s_0$ by $\kappa(s_0)$. Show that

$$|\kappa(s_0)| \geq \min_{s \in I}\left\{\frac{1}{\|\alpha(s) - \mathbf{p}\|}\right\}.$$

(**Hint 1**: recall the definition of relative extreme points in differential calculus.)

(**Hint 2**: Any relations between dot product and norm?)

6. Let $\{\gamma_t(u)\}$ be a family of closed smooth plane curves with $u \in [0, 2\pi)$ and $t \in I$ where I is open interval. Let $\gamma_t(s)$ be the arc-length parametrization of $\gamma_t(u)$ for each t. Suppose it satisfies the following heat equation (with notation $\gamma = \gamma(s,t) = \gamma_t(s)$):

$$\frac{\partial}{\partial t}\gamma = \frac{\partial^2}{\partial s^2}\gamma.$$

(a) (3 points) Explain why

$$\frac{\mathrm{d}s}{\mathrm{d}u} = |\gamma'(u)|.$$

By inverse function theorem we therefore have

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{1}{|\gamma'(u)|}.$$

(b) (4 points) Given $\frac{\partial}{\partial t}|\gamma'(u)| = -\kappa^2|\gamma'(u)|$, show that

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}f - \frac{\partial}{\partial s}\frac{\partial}{\partial t}f = \kappa^2\frac{\partial}{\partial s}f$$

for all smooth function $f(u,t)$.

(**Hint**: Start with $\dfrac{\partial}{\partial t}\dfrac{\partial f}{\partial s} = \dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial s}\dfrac{\partial f}{\partial u}\right)$)

(c) i. (2 points) Do you agree that $\frac{\partial \mathbf{N}}{\partial t}$ must be perpendicular to $\mathbf{N}$? Explain.

ii. (4 points) Combine all the above information, show for the Frenet frame $\{\mathbf{T}, \mathbf{N}\}$ of $\gamma_t(s)$:

$$\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial \kappa}{\partial s}\mathbf{N}, \quad \frac{\partial \mathbf{N}}{\partial t} = -\frac{\partial \kappa}{\partial s}\mathbf{T}.$$

7. (15 points (bonus)) This question is related to the brachistochrone problem. It examines students' ability to read (easy) mathematical text in modern geometry.

   Consider the following function J mapping a function L to a scalar quantity:

$$J[L] = \int_{x_i}^{x_f} L(x, y(x), y'(x))\, dx$$

   where $x \in [x_i, x_f], y : [x_i, x_f] \to \mathbb{R}$ and L is a function on $x, y(x)$ and $y'(x)$.

   Since J is a function on functions, extreme points of J are actually curves (as points in a function space). Suppose J attain its minimum with the curve $y = y_{\min}(x)$.

   We can find $y_{\min}(x)$ by the following steps:

   1. Denote the neighboring curve of $y_{\min}(x)$ by

$$y = y_{\min}(x) + \alpha\eta(x)$$

      where α is a parameter and $\eta(x)$ is a function satisfies $\eta(x_i) = \eta(x_f) = 0$. (Refer to Figure 1.)

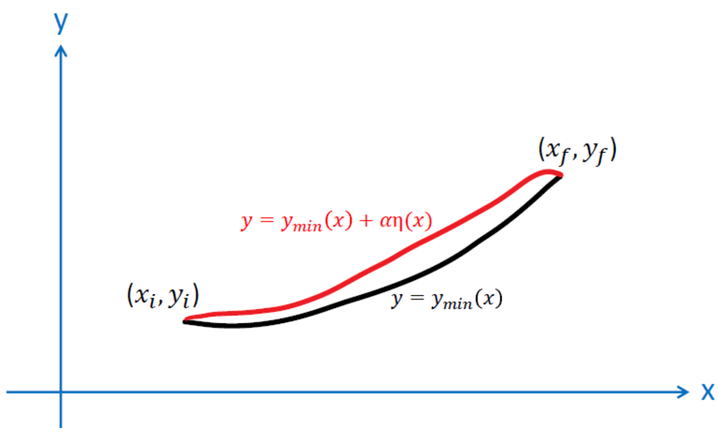

Figure 1: Illustration of Step 1

   2. We then have the relationship:

$$\begin{cases} y &= y_{\min}(x) + \alpha\eta(x) \\ y' &= y'_{\min}(x) + \alpha\eta'(x). \end{cases}$$

   3. Now we have

$$J = J(\alpha) = \int_{x_i}^{x_f} L(x, y_{\min}(x) + \alpha\eta(x), y'_{\min}(x) + \alpha\eta'(x))\mathrm{d}x.$$

      and the minimum of J occurs at $\alpha = 0$ (i.e. occurs at $y = y_{\min}(x)$). We also have

$$\frac{dJ}{d\alpha} = 0.$$

The questions are on the next page.

(a) i. It is given that $\frac{d}{d\alpha}\int_a^b h(x,\alpha)\,dx = \int_a^b \frac{\partial}{\partial\alpha} h(x,\alpha)\,dx$. Show that (3 marks)

$$\frac{dJ}{d\alpha} = \int_{x_i}^{x_f} \left(\frac{\partial L}{\partial y}\eta(x) + \frac{\partial L}{\partial y'}\eta'(x) \right) dx$$

ii. Hence or otherwise, show that when $\frac{dJ}{d\alpha} = 0$, we have (3 marks)

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0.$$

(**Hint**: If $\int_{\mathbb{R}} f\eta = 0$ for all function η with compact support in $\mathbb{R}$, then $f = 0$.)

iii. Further suppose $\frac{\partial L}{\partial x} = 0$, show that if $\frac{dJ}{d\alpha} = 0$, then there exists $C \in \mathbb{R}$ such that (3 marks)

$$y'\frac{\partial L}{\partial y'} - L = C$$

(**Hint**: Differentiate $L(x, y(x), y'(x))$ with respect to x by multivariable chain rule)

(b) Clive would like to ride on a slide $\mathbf{r} : [0, a] \to \mathbb{R}^2$ in a playground, which is defined by

$$\mathbf{r}(x(t)) = (x(t), -y(x(t)))$$

where $y : [0, a] \to \mathbb{R}$ satisfies initial conditions $y(0) = 0, y(a) = b$.
Let $\mathbf{g}$ be a real constant, the time required for Clive to finish a ride is given by

$$T = \int_0^a \sqrt{\frac{1 + [y'(x)]^2}{2\mathbf{g}y(x)}}\,dx$$

i. Let $a, c \in \mathbb{R}^+$ with $c > a$, evaluate the integral (2 marks)

$$\int_0^a \sqrt{\frac{y}{c-y}}\,dy$$

ii. Using (a), show that the time minimising curve is parameterised by (3 marks)

$$\mathbf{y}_{\min}(\theta) = (A(\theta - \sin\theta), A(1 - \cos\theta))$$

for some constant A.
(You are not required to compute A and state the exact range of θ)

iii. Name the curve in (b)(ii). (1 mark)

Remark. This delightful question comes from Nelson and Tommy, Clive is still on his way sliding down.